Propagation of dark solitons with higher-order effects in optical fibers

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In this paper, we analyze dark soliton propagation in nonlinear optical fibers with higher-order effects such as third order dispersion, self-steepening, and stimulated Raman scattering. We consider the Hirota equation and the higher-order nonlinear Schrödinger equation, and identify conditions for dark soliton propagation through Painlevé analysis. We also construct an explicit Lax pair, and Hirota bilinear form is used to generate one and two dark solitons.

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I. INTRODUCTION

Optical solitons have been the objects of extensive theoretical and experimental studies during the last three decades, because of their potential applications in long distance communication. The pioneering works of Hasegawa and Tappert [1], who predicted solitons theoretically, and Mollenauer, Stolen, and Gordon [2], who observed them experimentally, made solitons a realistic tool for this cause. The solitons, localized-in-time optical pulses, evolve from a nonlinear change in the refractive index of the material, known as Kerr effect, induced by the light intensity distribution. When the combined effects of the intensity-dependent refractive index nonlinearity and the frequency-dependent pulse dispersion exactly compensate for one another, the pulse propagates without any change in its shape, being self-trapped by the waveguide nonlinearity. The propagation of optical solitons in a nonlinear dispersive optical fiber is governed by the well-known completely integrable nonlinear Schrödinger equation which is of the form

$$iq_t \pm (1/2)q_{xx} + |q|^2 q = 0, \tag{1}$$

where q is the complex amplitude of the pulse envelope, x and t represent the spatial and temporal coordinates, and the $+$ or $-$ signs before the dispersive term denote the anomalous and normal dispersive regimes, respectively. In the anomalous dispersive regime, this equation possesses a bright soliton solution, and in the normal dispersive regime it possesses dark solitons. When compared with bright solitons, the investigations of dark solitons are inadequate. However, in recent years, the dark soliton has also attracted a lot of attention, and many innovative results have already appeared concerning this exciting topic.

The generation of dark solitons was first predicted by Hasegawa and Tappert [3] and Zakharov and Shabat [4], and experimentally demonstrated by Emplit *et al.* [5]. The bright soliton is a pulse on a zero-intensity background, while a dark soliton appears as an intensity dip in an infinitely extended constant background. Apart from the inverse intensity profile, an additional unique feature of a dark soliton is its specific phase profile. The dark-soliton phase chirp is a monotonic and odd function of the spatial coordinate. Recently, increased interest in dark spatial solitons has become connected with their possible application in optical logic devices [6] and waveguide optics as dynamic switches and junctions [7]. These applications are based on the fact that dark spatial solitons actually create waveguides in a self-defocusing medium. They are also considered for signal processing and communication applications because of their inherent stability [8]. In fact, the influence of noise and fiber loss on dark solitons is much lesser than that on bright solitons. Recently, Kivshar and Davies gave an extensive review article on dark solitons which discussed the above points and more [9].

Motivated by these facts, in this paper, we consider the theoretical aspects of dark soliton propagation in nonlinear Schrödinger (NLS)-type equations, namely, the Hirota equation and the higher-order NLS equation which include the higher-order effects such as third order dispersion, self-steepening, and stimulated inelastic scattering. We explicitly construct a Lax pair for the dark solitons in the NLS equation, and the other two higher-order equations and an extension to N-coupled systems are also discussed. In the case of higher-order systems, in addition to the already known bright soliton case, a case for dark solitons is identified through Painlevé analysis. The dark one- and two-soliton solutions are generated by means of Hirota's bilinear form and the significance of these solutions are discussed.

The paper is organized as follows. In Sec. II, we discuss the Lax pair for dark solitons in the NLS system. In Sec. III, we discuss the Painlevé analysis of the higher-order NLS (HNLS) equation through which cases of dark soliton systems are identified. In Sec. IV, we give a Lax pair for the Hirota equation and darks one- and two-soliton solutions are obtained for the Hirota equation through Hirota's bilinear form. We also construct the Lax pair for the HNLS equation in Sec. V. For this system, dark one- and two-soliton solutions are presented.

II. LAX PAIR FOR DARK SOLITONS IN THE NLS SYSTEM

The Lax pair assures the complete integrablity of a nonlinear system, and is especially used to obtain N-soliton solutions by means of inverse scattering transform method. In this paper, we follow the Ablowitz, Kaup, Newell, and Segur (AKNS) formalism to obtain the Lax pair. The linear eigenvalue problem for optical solitons in the NLS system can be constructed as follows:

$$\Psi_x = U\Psi,$$

$$\Psi_t = V\Psi,$$

where

$$\Psi = (\Psi_1\, \Psi_2)^T. \tag{2}$$

Here the Lax operators U and V are given in the forms

$$U = \begin{pmatrix} -i\lambda/2 & -\mu q \\ \mu q^* & i\lambda/2 \end{pmatrix},$$

$$V = \lambda^2 \begin{pmatrix} -i\bar{\mu}/2\mu & 0 \\ 0 & i\bar{\mu}/2\mu \end{pmatrix} + \lambda \begin{pmatrix} 0 & -\bar{\mu} q \\ -\bar{\mu} q^* & 0 \end{pmatrix} + \begin{pmatrix} i\mu\bar{\mu}|q|^2 & -i\bar{\mu} q_x \\ -i\bar{\mu} q_x^* & -i\mu\bar{\mu}|q|^2 \end{pmatrix}, \tag{3}$$

where λ is the eigenvalue parameter, and μ and $\bar{\mu}$ are constants whose choices make the resultant equation either for bright or dark solitons as indeed shown below.

Case (i): $\mu=\bar{\mu}=1$. For this case, the compatibility condition $U_t - V_x + [U,V] = 0$ gives the nonlinear Schrödinger equation for bright solitons of the form

$$iq_t + q_{xx} + 2|q|^2 q = 0. \tag{4}$$

Case (ii): $\mu=i$ and $\bar{\mu}=-i$. For this case, the compatibility condition gives the nonlinear Schrödinger equation for dark solitons:

$$iq_t - q_{xx} + 2|q|^2 q = 0. \tag{5}$$

Thus, in this section, we discuss a single system of Lax pair for both bright and dark solitons for the nonlinear Schrödinger equation. In the case of bright solitons, the Bäcklund transformation and soliton solutions are well known [10]. However, for the dark solitons, we are not able to obtain Bäcklund transformation and soliton solutions yet. Hence we proceed to determine the conditions for dark soliton propagation in higher-order systems.

III. PAINLEVÉ ANALYSIS OF THE NLS EQUATION WITH HIGHER-ORDER EFFECTS

Even though the NLS equation explains pulse propagation in a nonlinear optical fiber, it has its own limitations. For example, when the optical pulse is of the order of femtoseconds, the NLS equation becomes inadequate, as higher order effects like third order dispersion (TOD), self-steepening (SS), and stimulated Raman scattering (SRS) should be included. In such a case, the governing equation is the one known widely as the higher-order NLS equation, first derived by Kodama and Hasegawa [11]. The effect of these effects in uncoupled and coupled systems for bright solitons is well explained in many papers [12–14]. Inelastic Raman scattering is due to the delayed response of the medium, which forces the pulse to undergo a frequency shift which is known as a self-frequency shift. The effect of self-steepening is due to the intensity-dependent group velocity of the optical pulse, which gives the pulse a very narrow width in the course of propagation. Because of this, the peak of the pulse will travel more slowly than the wings. Recently, Painlevé analysis for this equation has been carried out by many authors, but they identified only the bright soliton case; the dark soliton case was not reported [15–18]. Here it should be mentioned that the authors of Ref. [18] anticipated that Painlevé analysis of the HNLS equation could also be extended to the dark case. Also, a thorough analysis of the Sasa-Satsuma equation was performed in the case of bright solitons by using the inverse scattering transform method [19] and by the Riemann problem method [20]. By fully exploiting the symmetry properties of the scattering matrix elements, the most general one-parameter single-soliton solution, the four parameter breather soliton solution and the most general N-soliton solution of the perturbed nonlinear Schrödinger equation in the bright soliton case were given by the authors of Ref. [18] in these references.

The effect of third order dispersion was discussed by Kivshar and Afanasjev, who showed that near the zero point of the group velocity dispersion, dark solitons exist as humps instead of dips. It was proved that the solitary wave acts as a source generating trailing oscillations, which with the leading front propagates with the group velocity V_g [21]. When we take third order dispersion and self-steepening together into account with the group velocity and self-phase modulation terms of the NLS system, the governing equation is known as the Hirota equation, whose bright soliton properties were analyzed by many authors [22,23]. On the other hand, if we include the stimulated inelastic scattering together with these two effects, we would obtain the higher-order NLS equation. For the known integrable systems, Radhakrishnan and Lakshmanan [24] considered both bright and dark soliton propagation in higher-order NLS systems. In this paper, we carry out the Painlevé analysis to find out new integrability conditions for the case of dark solitons. The HNLS equation is given in the form

$$q_t = \pm iq_{xx} + 2i|q|^2 q + \varepsilon[q_{xxx} + \alpha_1(|q|^2 q)_x + \alpha_2 q(|q|^2)_x] \tag{6}$$

where q is the slowly varying amplitude of the pulse envelope, and α_1 and α_2 are arbitrary constants and the $+$ sign corresponds to the anomalous dispersive regime and the $-$ sign to normal dispersive regime. The parameter ϵ represents the relative width of the spectrum that arises due to quasi-monochromocity, and it is assumed that $0<\epsilon<1$. As the bright soliton versions of the above equations are well studied, in this paper, we consider only the dark soliton version of Eq. (6). To identify the new integrable systems, we follow the Weiss, Tabor, and Carnevale (WTC) procedure [25] to carry out the Painlevé analysis, according to which a given partial differential equation (PDE) is integrable, if its solutions are single valued about the movable singularity manifold. This method requires the following steps to prove the integrability: (i) determination of the leading orders of Laurent series, (ii) identification of the powers at which the arbitrary functions can enter into the Laurent series called resonances and (iii) verification of the existence of sufficient number of arbitrary functions at the resonance values without the introduction of a movable critical manifold and connection with complete integrable properties. Throughout this analysis, we used the Kruskal's reduced manifold ansatz.

In order to carry out the Painlevé analysis, let us assume $q=a$ and $q^*=b$. Equation (6) becomes

$$a_t=-ia_{xx}+2ia^2b+\varepsilon[a_{xxx}+\alpha_1(a^2b)_x+\alpha_2 a(ab)_x], \tag{7a}$$

$$b_t=ib_{xx}-2ib^2a+\varepsilon[b_{xxx}+\alpha_1(b^2a)_x+\alpha_2 b(ab)_x]. \tag{7b}$$

To determine the leading order behavior, we expand

$$a\approx a_0\phi^m \quad \text{and} \quad b\approx b_0\phi^n, \tag{8}$$

where m and n are negative integers. Substituting Eq. (8) into Eqs. (7) and equating the dominant terms, we obtain

$$m=n-1 \quad \text{and} \quad a_0b_0=-6/(3\alpha_1+2\alpha_2). \tag{9}$$

To find the resonances, we substitute

$$a=a_0\phi^{-1}+a_j\phi^{j-1} \quad \text{and} \quad b=b_0\phi^{-1}+b_j\phi^{j-1}. \tag{10}$$

Collecting the coefficients of φ^{j-4} and solving the resultant determinant, the resonances are obtained as

$$j=-1,0,3,4,3\pm\frac{2\alpha_2}{\sqrt{-3\alpha_1\alpha_2-2\alpha_2^2}}. \tag{11}$$

The resonance at $j=-1$ corresponds to the arbitrariness of the singular manifold and the arbitrariness at $j=0$ is verified from Eq. (9), which shows that either a_0 or b_0 is arbitrary. From the resonance analysis, it can be seen that there are two possible cases for the resonances to be integers. They are

$$\begin{aligned}&\text{case (i):}\quad \alpha_1=-\alpha_2,\\ &\text{case (ii):}\quad \alpha_1=-2\alpha_2.\end{aligned} \tag{12}$$

At this juncture, it is interesting to note that case (i) has the resonances

$$j=-1,0,1,3,4,5 \tag{13}$$

for both $\alpha_1=-\alpha_2=6$ and $\alpha_1=-\alpha_2=-6$.

The former case corresponds to well-known bright solitons for the Hirota equation, and the latter one corresponds to dark soliton solutions. From arbitrary analysis, we also found that Eq. (6) admits sufficient number of arbitrary functions at the resonance values as proved below.

On the other hand, case (ii) has the resonances

$$j=-1,0,2,3,4,4 \tag{14}$$

for both $\alpha_1=-2\alpha_2=6$ and $\alpha_1=-2\alpha_2=-6$. The former case corresponds to the well-known bright solitons for the HNLS equation first given by Sasa and Satsuma [12], and the latter case corresponds to the dark soliton solutions.

The Hirota equation corresponding to dark solitons can be given as

$$q_t=-iq_{xx}+2i|q|^2q+\varepsilon[q_{xxx}+\alpha_1(|q|^2q)_x-\alpha_1 q(|q|^2)_x]. \tag{15}$$

In order to check the existence of a sufficient number of arbitrary functions at other resonance values, we substitute the full Laurent series in Eqs. (7) with $\alpha_2=-\alpha_1$. From the coefficients of $(\varphi^{-3},\varphi^{-3})$, it can be shown that

$$a_0b_1=-b_0a_1 \quad \text{and} \quad \alpha_1=-6. \tag{16}$$

Equation (16) clearly shows that either a_1 or b_1 is arbitrary, which corresponds to the resonance at $j=1$. Also, the value of α_1 shows that it corresponds to a case of integrability for which dark solitons may occur. Collecting the coefficients of $(\varphi^{-2},\varphi^{-2})$, we obtain values for a_2 and b_2 as follows:

$$\begin{aligned}a_2&=\frac{-1}{6\varepsilon b_0}[\phi_t+2ia_0b_1+6\varepsilon a_1b_1],\\ b_2&=\frac{-1}{6\varepsilon a_0}[\phi_t-2ib_0a_1+6\varepsilon a_1b_1].\end{aligned} \tag{17}$$

Similarly from other coefficients of φ, one can prove the existence of sufficient number of arbitrary functions without the introduction of any movable critical manifold. Hence it can be concluded that the system of Hirota equation for dark solitons passes the Painlevé analysis, and is expected to be integrable.

Now let us consider the arbitrary analysis of the Sasa-Satsuma case for dark solitons in which $\alpha_1=-2\alpha_2$, and the resonances are given by Eq. (14). The HNLS equation for dark solitons is given in the following form:

$$q_t=-iq_{xx}+2i|q|^2q+\varepsilon\left[q_{xxx}+\alpha_1(|q|^2q)_x-\frac{\alpha_1}{2}q(|q|^2)_x\right]. \tag{18}$$

We recall that the resonances for this case are given by Eq. (14). In order to check the existence of a sufficient number of arbitrary functions at other resonance values, we substitute the full Laurent series in Eqs. (7) with $2\alpha_2=-\alpha_1$. From the coefficients of $(\varphi^{-3},\varphi^{-3})$, it can be shown that

$$a_1=\frac{2i(a_0b_0-1)}{\alpha_1\varepsilon b_0}, \tag{19a}$$

$$b_1=\frac{-2i(a_0b_0-1)}{\alpha_1\varepsilon a_0}. \tag{19b}$$

Similarly, from the coefficient of $(\varphi^{-2},\varphi^{-2})$, we can show that either a_2 or b_2 is arbitrary, which corresponds to the resonance at $j=2$. From higher powers of φ, one can show that in order to prove the existence of a sufficient number of arbitrary functions, $\alpha_1=-6$ and hence $\alpha_2=-3$. Thus it is concluded that the dark soliton cases for both Hirota and HNLS equations are expected to be integrable only for the particular values of α_1 and α_2 viz $\alpha_1=-6$ and $\alpha_2=-3$.

It is interesting to note that the constraints given in Eqs. (12)–(14) obtained from Painlevé analysis are similar to the one obtained for the bright soliton case, except for sign changes in the parameters of higher-order effects. The presence of bright optical solitons with higher-order effects has been experimentally verified, and the influence of these effects well studied. However, the dark solitons with these higher order effects is of only theoretical interest now. However, we believe that the constraint given by Eqs. (12)–(14) is experimentally realizable, as we are able to obtain dark one- and two-soliton solutions for the integrable versions of these systems. Thus Painlevé analysis proves to be a valuable tool for obtaining the constraints of the parameters for the existence of dark solitons. Having proved the integrability of dark Hirota and HNLS systems through Painlevé analysis, now we move on to establish the complete integrability properties of these systems, such as the Lax pair, Hirota's bilinear form, and soliton solutions.

IV. LAX PAIR FOR DARK HIROTA SYSTEM

The linear eigenvalue problem for dark solitons of the Hirota system can be constructed as follows:

$$\Psi_x=U\Psi,$$

$$\Psi_t=V\Psi,$$

where

$$\Psi=(\Psi_1 \quad \Psi_2)^T. \tag{20}$$

Here U and V can be given in the forms

$$U=\begin{pmatrix} -i\lambda/2 & -iq \\ iq^* & i\lambda/2 \end{pmatrix},$$

$$V=\lambda^3\begin{pmatrix} i\varepsilon/2 & 0 \\ 0 & -i\varepsilon/2 \end{pmatrix}+\lambda^2\begin{pmatrix} i/2 & i\varepsilon q \\ -i\varepsilon q^* & -i/2 \end{pmatrix}+\lambda\begin{pmatrix} i\varepsilon|q|^2 & iq-\varepsilon q_x \\ -iq^*-\varepsilon q_x^* & -i\varepsilon|q|^2 \end{pmatrix}$$
$$+\begin{pmatrix} i|q|^2+\varepsilon(qq_x^*-q^*q_x) & 2i\varepsilon|q|^2q-q_x-i\varepsilon q_{xx} \\ -2i\varepsilon|q|^2q-q_x^*+i\varepsilon q_{xx} & -i|q|^2+\varepsilon(-qq_x^*+q^*q_x) \end{pmatrix}. \tag{21}$$

The compatibility condition $U_t-V_x+[U,V]=0$ gives rise to Eq. (15).

Since we are able to obtain the Lax pair for the dark soliton version of the Hirota equation, we conclude that it is possible to perform the inverse scattering transform method for this equation to obtain N dark soliton solutions. We give the Hirota bilinear form and the one-soliton solution for this case below.

Bilinearization and dark soliton solutions

Hirota's bilinear method [26] is one of the most direct and elegant methods available to generate multisoliton solutions of nonlinear PDE's. In order to obtain dark soliton solutions for the Hirota equation, we rewrite Eq. (15) in the more conventional form

$$iq_t-q_{xx}+2|q|^2q-i\varepsilon\{q_{xxx}-6|q|^2q_x\}=0. \tag{22}$$

To avoid mathematical complexities, it is rather convenient to transform this equation to a simpler form, so that we may be able to generate multisoliton solutions. We make the following transformations to convert the Hirota equation to complex modified KdV (cmKdV) equations:

$$q(x,t)=Q(Z,T)\exp\left[i\left(\frac{Z}{3\varepsilon}-\frac{T}{27\varepsilon^2}\right)\right],$$

$$T=t,\quad Z=x+\frac{t}{3\varepsilon}. \tag{23}$$

Using the above transformations in Eq. (22), the resultant complex modified KdV equation is obtained in the form

$$Q_T-\varepsilon\{Q_{ZZZ}-6|Q|^2Q_Z\}=0. \tag{24}$$

Next we consider the Hirota bilinear transformation

$$Q=G/F, \tag{25}$$

where $G(Z,T)$ is a complex function and $F(Z,T)$ is a real function. Using Eq. (25), we obtain the decoupled forms of the bilinearized cmK-dV equation as follows:

$$(D_T-\varepsilon D_Z^3-3\varepsilon\lambda D_Z)G\cdot F=0, \tag{26a}$$

$$(D_Z^2+\lambda)FF=-2|G|^2, \tag{26b}$$

where λ is a constant to be determined, and the Hirota bilinear operators D_x and D_t are defined as

$$D_x^m D_t^n G(x,t)\cdot F(x,t)=\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial t}-\frac{\partial}{\partial t'}\right)^n \times G(x,t)\cdot F(x',t')\bigg|_{x=x',t=t'}. \tag{27}$$

Further, we assume that

$$G=g_0(1+\chi g_1) \quad \text{and} \quad F=1+\chi f_1, \tag{28}$$

where g_0 is a complex constant, and g_1 and f_1 are real functions. Substituting Eq. (28) into Eq. (26), and collecting the coefficients of χ^0, we obtain

$$\lambda=-2|g_0|^2. \tag{29}$$

The coefficients of χ and χ^2 lead to the following equations:

$$(D_T-\varepsilon D_z^3-3\varepsilon\lambda D_Z)(1\cdot f_1+g_1\cdot 1)=0, \tag{30a}$$

$$(D_Z^2+\lambda)(1\cdot f_1+f_1\cdot 1)=-4|g_0|^2 g_1, \tag{30b}$$

$$(D_T-\varepsilon D_Z^3-3\varepsilon\lambda D_Z)(g_1\cdot f_1)=0, \tag{30c}$$

$$(D_Z^2+\lambda)(f_1 f_1)=-4|g_0|^2 g_1^2. \tag{30d}$$

It can be easily shown that Eqs. (30) can be solved if we assume that

$$g_1=-f_1=-\exp[\omega_1 T+c_1 Z+\xi_i^{(0)}], \tag{31}$$

where

$$\omega_1=\varepsilon c_1(c_1^2+3\lambda), \qquad c_1^2=-2\lambda=4|g_0|^2. \tag{32}$$

Using Eqs. (32), (31), and (25), we can obtain the dark one-soliton solution in the form

$$Q=g_0\tanh\left[\frac{1}{2}\left\{c_1\left(Z-\frac{c_1^2\varepsilon T}{2}\right)+\xi_1^{(0)}\right\}\right]. \tag{33}$$

By using transformations (23), we can obtain the dark one-soliton solution of the Hirota equation. It is clear that the higher-order effects TOD and SS effects affect the velocity of the dark soliton, yet they propagate without any change in their shape and intensity. Next, we move on to the construction of two-soliton solutions of the Hirota equation. To obtain the two-soliton solution, we assume the following forms for G and F:

$$G=g_0(1+\chi g_1+\chi^2 g_2) \quad \text{and} \quad F=1+\chi f_1+\chi^2 f_2. \tag{34}$$

The coefficients of χ^0 lead to Eq. (29). From the coefficient of χ, we obtain

$$(D_T-\varepsilon D_Z^3-3\varepsilon\lambda D_Z)(1\cdot f_1+g_1\cdot 1)=0, \tag{35a}$$

$$(D_Z^2+\lambda)(1\cdot f_1+f_1\cdot 1)=-4|g_0|^2 g_1. \tag{35b}$$

To solve these equations, we assume

$$g_1=P_1\exp[\xi_1]+P_2\exp[\xi_2]$$

and

$$f_1=\exp[\xi_1]+\exp[\xi_2], \tag{36a}$$

where $\xi_1=\omega_1 T+c_1 Z+\xi_1^{(0)}$ and $\xi_2=\omega_2 T+c_2 Z+\xi_2^{(0)}$ with

$$\omega_1=\varepsilon c_1^3+3\varepsilon\lambda c_1 \quad \text{and} \quad \omega_2=\varepsilon c_2^3+3\varepsilon\lambda c_2. \tag{36b}$$

The values of P_1 and P_2 are found to be

$$P_1=\frac{2|g_0|^2-c_1^2}{2|g_0|^2}, \qquad P_2=\frac{2|g_0|^2-c_2^2}{2|g_0|^2}. \tag{36c}$$

The coefficients of χ^2 lead to the following equations:

$$(D_T-\varepsilon D_Z^3-3\varepsilon\lambda D_Z)(1\cdot f_2+g_1 f_1+g_2\cdot 1)=0, \tag{37a}$$

$$(D_Z^2+\lambda)(1\cdot f_2+f_1 f_1+f_2\cdot 1)+2(2|g_0|^2 g_2+|g_0|^2 g_1^2)=0. \tag{37b}$$

It can be shown that the above system of equations can be satisfied if we assume

$$g_2=A_{12}P_1P_2\exp[\xi_1+\xi_2] \quad \text{and} \quad f_2=A_{12}\exp[\xi_1+\xi_2] \tag{38a}$$

The value of A_{12} is found to be

$$A_{12}=\frac{(P_2-P_1)\{-(\omega_2-\omega_1)+\varepsilon(c_2-c_1)^3+3\varepsilon\lambda(c_2-c_1)\}}{(1-P_1P_2)\{-(\omega_2+\omega_1)+\varepsilon(c_2+c_1)^3+3\varepsilon\lambda(c_2+c_1)\}}. \tag{38b}$$

From the values of g_1, g_2, g_3, f_1, and f_2, one can obtain dark two-soliton solutions of the Hirota equation. In Fig. 1, this two-soliton solution is plotted. From detailed investigations, we find that dark two-soliton solutions behave in an elastic manner characteristic of all soliton solutions. They retain their shapes after collision with only a slight change in their phase. Also, like all dark solitons, they appear to repel each other, and hence there is no possibility of forming a bound state between them. This important feature is an attractive factor that makes dark solitons a preferred tool, instead of bright solitons, in long-distance communications. Our next aim is to find the dark one- and two-soliton solutions for the HNLS system.

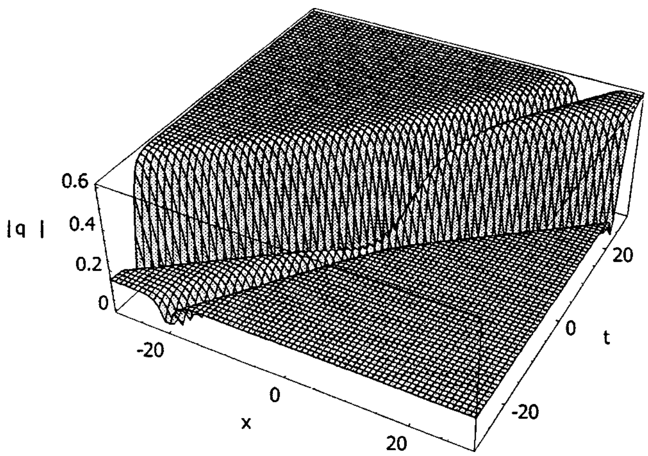

FIG. 1. Dark two-soliton solution of the Hirota system.

V. LAX PAIR FOR DARK HNLS SYSTEM

The integrable version of the dark HNLS equation takes the form

$$q_t = -iq_{xx} + 2i|q|^2 q + \varepsilon[q_{xxx} - 6q_x|q|^2 - 3q(|q|^2)_x]. \tag{39}$$

As described in the Sec. IV, we present the linear eigenvalue problem and the Lax pair in the same manner:

$$\Psi_x = U\Psi,$$

$$\Psi_t = V\Psi,$$

where

$$\Psi = (\Psi_1 \;\; \Psi_2 \;\; \Psi_3)^T. \tag{40}$$

The Lax operators U and V can be given in the forms

$$U = \begin{pmatrix} -i\lambda/2 & -iq & -ir \\ iq^* & i\lambda/2 & 0 \\ ir^* & 0 & i\lambda/2 \end{pmatrix}, \tag{41a}$$

$$V = \lambda^3 \begin{pmatrix} i\varepsilon/2 & 0 & 0 \\ 0 & -i\varepsilon/2 & 0 \\ 0 & 0 & -i\varepsilon/2 \end{pmatrix} + \lambda^2 \begin{pmatrix} i/2 & i\varepsilon q & i\varepsilon r \\ -i\varepsilon q^* & -1/2 & 0 \\ -i\varepsilon r^* & 0 & -i/2 \end{pmatrix} + \lambda \begin{pmatrix} i\varepsilon(|q|^2+|r|^2) & -\varepsilon q_x + iq & -\varepsilon r_x + ir \\ -\varepsilon q_x^* - iq^* & -i\varepsilon|q|^2 & -iq^* r \\ -\varepsilon r_x^* - ir^* & -ir^* q & -i\varepsilon|r|^2 \end{pmatrix} + \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix},$$

where

$$\begin{aligned} A_{11} &= i|q|^2 + i|r|^2 + \varepsilon(qq_x^* - q_x q^* + rr_x^* - r_x r^*), \\ A_{12} &= -q_x - i\varepsilon q_{xx} + 2i\varepsilon(q|q|^2 + r|r|^2), \\ A_{13} &= -r_x - i\varepsilon r_{xx} + 2i\varepsilon(r|q|^2 + r|r|^2), \\ A_{21} &= -q_x^* + i\varepsilon q_{xx}^* - 2i\varepsilon(q^*|q|^2 + q^*|r|^2), \\ A_{22} &= -i|q|^2 + \varepsilon(q^* q_x - qq_x^*), \\ A_{23} &= -iq^* r + \varepsilon(q^* r_x - rq_x^*), \\ A_{31} &= -r_x^* + i\varepsilon r_{xx}^* - 2i\varepsilon(r^*|q|^2 + r^*|r|^2), \\ A_{32} &= -ir^* q + \varepsilon(r^* q_x - qr_x^*), \\ A_{33} &= -i|r|^2 + \varepsilon(r^* r_x - rr_x^*), \end{aligned} \tag{41b}$$

with

$$r = e^{i\Theta} q^* \quad \text{and} \quad \Theta(x,t) = \tfrac{2}{3}(x + \tfrac{2}{9}t). \tag{41c}$$

The compatibility condition $U_t - V_x + [U,V] = 0$ gives rise to Eq. (39). The construction of Lax pair confirms that the dark HNLS equation is indeed completely integrable. The next logical step would be to extend our above results to N-coupled systems to analyze the N-field propagation. This work is under progress and the results will be published soon. Though we are able to obtain the Lax pair for dark soliton systems, we could not yet obtain dark soliton solutions from it through standard methods like Bäcklund transformation. Hence, in the next subsection, we use Hirota's bilinear technique to derive the dark soliton solutions.

Bilinearization and dark soliton solutions

We follow the same method used for the Hirota equation in the previous sections to obtain dark soliton solutions for HNLS equation. First we transform Eq. (39) to a CmK-dV equation using Eqs. (24) as follows:

$$Q_T - \varepsilon[Q_{ZZZ} - 6|Q|^2 Q_Z - 3Q(|Q|^2)_Z] = 0. \tag{42}$$

The decoupled bilinear forms of Eq. (42) are given as

$$(D_T - \varepsilon D_Z^3 + 3\varepsilon\lambda D_Z) G \cdot F = 0, \tag{43a}$$

$$(D_Z^3 - \lambda) F \cdot F = -4|G|^2, \tag{43b}$$

$$D_Z G^* \cdot G = 0. \tag{43c}$$

To obtain one-soliton solutions, we assume

$$G = g_0(1 + \chi g_1) \quad \text{and} \quad F = 1 + \chi f_1, \tag{44}$$

where, g_0 is a complex constant, and g_1 and f_1 are real functions. Substituting Eq. (44) into Eq. (43), and collecting the coefficients of χ^0, we obtain

$$\lambda = 4|g_0|^2. \tag{45}$$

The coefficients of χ lead to the following equations:

$$(D_T-\varepsilon D_Z^3+3\varepsilon\lambda D_Z)(1\cdot f_1+g_1\cdot 1)=0, \tag{46a}$$

$$(D_Z^2-\lambda)(1\cdot f_1+f_1\cdot 1)+8|g_0|^2 g_1=0, \tag{46b}$$

$$D_Z(1\cdot g_1+g_1\cdot 1)=0. \tag{46c}$$

The coefficients of χ^2 lead to the following equations:

$$(D_T-\varepsilon D_Z^3+3\varepsilon\lambda D_Z)(g_1\cdot f_1)=0, \tag{47a}$$

$$(D_Z^2-\lambda)(f_1\cdot f_1)+4|g_0|^2 g_1^2=0, \tag{47b}$$

$$D_Z g_1 g_1=0. \tag{47c}$$

Equations (46) and (47) suggest that they can be solved if we assume

$$g_1=-f_1=-\exp[\omega_1 T+c_1 Z+\xi_1^{(0)}], \tag{48a}$$

where

$$\omega_1=\varepsilon c_1(c_1^2-3\lambda),$$

$$c_1^2=2\lambda=8|g_0|^2. \tag{48b}$$

Using Eqs. (48) and (44), the dark one-soliton solution of cmK-dV equation is obtained as

$$Q=g_0\tanh\left[\frac{1}{2}\left\{c_1\left(Z-\frac{c_1^2\varepsilon T}{2}\right)+\xi_1^{(0)}\right\}\right]. \tag{49}$$

Using transformation (23), we can easily obtain the corresponding dark one-soliton solution of the HNLS equation (39). It is clear that the higher-order effects TOD, SS, and SRS affect the velocity of the dark soliton, yet they propagate without any change in their shape and intensity. In this context, it should be mentioned that Mel'nikov *et al.* obtained the most general single soliton solution for Eq. (42), showing the dark-gray to dark-black bifurcation [27,28]. The dark soliton solution given in Eq. (49) can also be obtained from the most general solution, which exhibits either a single dip or two dips of equal width, given in these references with appropriate limit, viz by pushing one dip to infinity. It is interesting to note that the mKdV equation analyzed here is of interest not only in fiber optics. The twin hole dark solitary waves in nonintegrable systems were found in various physical settings such as the propagation of terahertz electromagnetic pulses in media characterized by the simultaneous presence of second and third order nonlinearities [29] and the parametric interaction in diffractive quadratic nonlinear media [30,31].

From here, we proceed to the next step of obtaining dark two-soliton solutions, for which we assume

$$G=g_0(1+\chi g_1+\chi^2 g_2) \quad \text{and} \quad F=1+\chi f_1+\chi^2 f_2. \tag{50}$$

where g_0 is a complex constant, and g_1, g_2, g_3, f_1, and f_2 are real functions. The coefficients of χ^0 lead to Eq. (45). From the coefficient of χ, we obtain

$$(D_T-\varepsilon D_Z^3+3\varepsilon\lambda D_Z)(1\cdot f_1+g_1\cdot 1)=0, \tag{51a}$$

$$(D_Z^2-\lambda)(1\cdot f_1+f_1\cdot 1)+8|g_0|^2 g_1=0. \tag{51b}$$

To solve these equations, we assume

$$g_1=P_1\exp[\xi_1]+P_2\exp[\xi_2] \quad \text{and}$$
$$f_1=\exp[\xi_1]+\exp[\xi_2], \tag{52a}$$

where $\xi_1=\omega_1 T+c_1 Z+\xi_1^{(0)}$ and $\xi_2=\omega_2 T+c_2 Z+\xi_2^{(0)}$, with

$$\omega_1=\varepsilon c_1^3-3\varepsilon\lambda c_1 \quad \text{and} \quad \omega_2=\varepsilon c_2^3-3\varepsilon\lambda c_2. \tag{52b}$$

The values of P_1 and P_2 are found to be

$$P_1=\frac{4|g_0|^2-c_1^2}{4|g_0|^2},$$

$$P_2=\frac{4|g_0|^2-c_2^2}{4|g_0|^2}. \tag{53}$$

The coefficients of χ^2 lead to the following equations:

$$(D_T-\varepsilon D_Z^3+3\varepsilon\lambda D_Z)(1\cdot f_2+g_1\cdot f_1+g_2\cdot 1)=0, \tag{54a}$$

$$(D_Z^2-\lambda)(1\cdot f_2+f_1\cdot f_1+f_2\cdot 1)+2(2|g_0|^2 g_2+|g_0|^2 g_1^2)=0. \tag{54b}$$

It can be shown that the above system of equations can be satisfied if we assume

$$g_2=A_{12}P_1P_2\exp[\xi_1+\xi_2] \quad \text{and} \quad f_2=A_{12}\exp[\xi_1+\xi_2]. \tag{55a}$$

The value of A_{12} is found to be

$$A_{12}=\frac{(P_2-P_1)\{-(\omega_2-\omega_1)+\varepsilon(c_2-c_1)^3-3\varepsilon\lambda(c_2-c_1)\}}{(1-P_1P_2)\{-(\omega_2+\omega_1)+\varepsilon(c_2+c_1)^3-3\varepsilon\lambda(c_2+c_1)\}}. \tag{55b}$$

The dark two-soliton solution for the HNLS equation can be obtained by using the expressions of g_1, g_2, f_1, and f_2. In Fig. 2, this two-soliton solution is plotted. The dark two-soliton solution behaves in an elastic manner characteristic of all soliton solutions. They retain their shape after collision only with a slight change in their phase. Also, like all dark solitons, they appear to repel each other. Thus for the first time, to our knowledge we have reported on a dark two-soliton solution of an integrable HNLS system.

VI. CONCLUSION

In this paper, we have discussed the dark solitons of NLS, the Hirota equation, and the HNLS equation. For the NLS

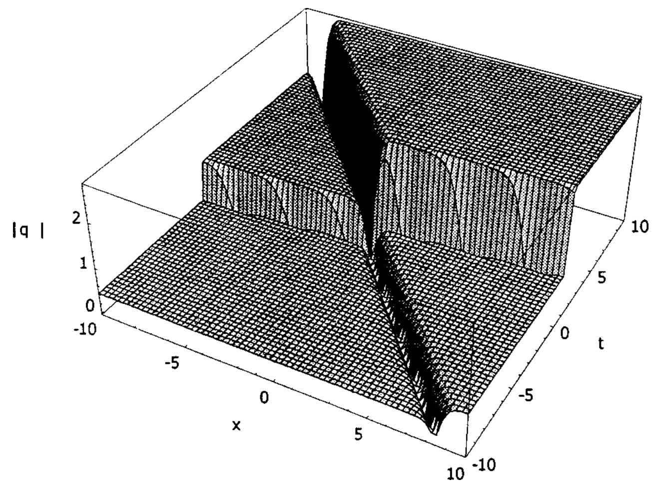

FIG. 2. Dark two-soliton solution of the HNLS system.

system, we have given the Lax pair for the dark soliton using the AKNS formalism. For the Hirota system, we have obtained dark soliton conditions using the Painlevé analysis. It was found that, the system admits dark soliton propagation when the coefficient of self-steepening is negative, -6 to be precise. The integrability of the above equation was also proved by the specific Lax pair. This clearly indicates the existence of dark solitons similar to the dark soliton propagation in NLS systems in the normal dispersion regime. Indeed, the dark one and two-soliton solutions were found by the Hirota bilinear method, and the solutions were plotted. Similar results were produced for the HNLS equation. In this case, the system admits dark soliton propagation when the coefficients of both self-steepening and inelastic Raman scattering are negative, -6 and -3, respectively.

It is seen that the higher-order effects TOD, SS, and SRS affect the velocity of the dark soliton, however, they propagate without any change in their shape and intensity. The dark two-soliton solution behaves in an elastic manner characteristic of all soliton solutions. The solitons retain their shape after collision only with a slight change in their phase. Also, like all dark solitons, they appear to repel each other, and hence there is no possibility of forming a bound state between them. This important feature is an attractive factor that makes dark solitons a preferred tool, instead of bright solitons, in long-distance communications. The bright soliton solutions for the nonlinear Schrödinger equation with higher-order effects are well known. What we have attempted in this paper is to establish dark soliton propagation, which is not well understood in systems with higher-order effects. Hence we have not tried to analyze gray soliton solutions. Also, dark solitons are more important than gray solitons from the point of view of practical applications such as optical communication, etc. Hence, we conclude that dark solitons can propagate in higher-order NLS systems under suitable physical conditions, as predicted in this paper by using an integrability analysis, and that they are experimentally realizable. Due to the superior nature of dark solitons when compared with bright solitons, viz. stability, repulsive nature, etc. we believe that these dark solitons are more favorable for the use of long-distance communication. The extension of the above equations to *N*-coupled system is under progress, and the results will be published soon.

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